A CLASSIFICATION, UP TO HYPERBOLICITY, OF GROUPS GIVEN BY 2 GENERATORS AND ONE RELATOR OF LENGTH 8

O. BOGOPOLSKI, N. BUSKIN AND A. BUTURLAKINN

ABSTRACT. We classify, up to hyperbolicity, all group given by a presentation with 2 generators and one cyclically reduced relator of length 8. In case of groups satisfying the small cancellation conditions $C(p)\,\&\,T(q)$ with parameters $(p,q)\in\{(3,6),(4,4),(6,3)\}$, we use the results and a technique of S. Ivanov and P. Schupp from [2].

1. INTRODUCTION

There are two difficult problems, concerning hyperbolic one relator groups:

1) Is it true that a one relator group is hyperbolic if and only if it does not contain any Baumslag – Solitar group $BS(n,m)=\langle a,b\,|\,a^{-1}b^na=b^m\rangle$, $n,m\neq 0$? (this is the conjecture of S. M. Gersten).
2) Does there exist an algorithm which, given a finite presentation with one relator, decides whether the corresponding group is hyperbolic?

It is convenient to investigate these problems in classes of groups, satisfying the small cancellation conditions $C(p)\,\&\,T(q)$ (see definitions in Sect. 4). Let G be a finitely presented group satisfying the conditions $C(p)\,\&\,T(q)$.
In [3] S. M. Gersten and H. Short proved that if $\frac{1}{p}+\frac{1}{q}<\frac{1}{2}$, then G is hyperbolic, and in [4] they proved that if $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, then G is automatic. The last condition means that $(p,q)\in\{(3,6),(4,4),(6,3)\}$). In this case the group need not be hyperbolic. For example, the free abelian group of rank 2 admits the presentation $\langle a,b\,|\,[a,b]\rangle$ which satisfies the $C(4)\,\&\,T(4)$-condition.

This paper was prepared while the first-named author was a visitor at Centre de Recerca Matemàtica in Barcelona. We thank the CRM for its hospitality. The research of the first-named author was partially supported by the grant Complex Integration Projects of SBRAS N 1.9, and by the INTAS grant N 03-51-3663.

In [2], S. Ivanov and P. Schupp gave a criterion of hyperbolicity of G in case $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$. While this criterion works good in many partial cases, it does not give an efficient algorithm for detecting the hyperbolicity.

The situation becomes more difficult, when $\frac{1}{p}+\frac{1}{q}>\frac{1}{2}$. There are hyperbolic and nonhyperbolic one relator groups which satisfy this condition. (It would be interesting to know their asymptotic proportion.)

The purpose of this paper is to initiate a research towards solving the above stated problems. We consider completely a very partial (but nontrivial) case, when G is a group given by a presentation $\langle a,b\,|\,R\rangle$, where R is a cyclically reduced word of length 8 over the alphabet $\{a,b\}^{\pm}$. There are 6564 such presentations. We consider them up to equivalence modulo permutations and inversions of a,b, and modulo cyclic permutations and inversions of R. There are 83 such equivalence classes. We classify them according to $C(p)\,\&\,T(q)$-conditions and with respect to the hyperbolicity (see Section 6 and the table in Section 8). It turns out that the groups in 32 classes are hyperbolic and in the rest 51 classes are not.

In establishing (non)hyperbolicity, we use the diagram criterion of Ivanov and Schupp [2] (which is explained in Sections 2-4) and theorems of Section 5. In Section 7 we prove that for a finite alphabet X of cardinality k the number of cyclically reduced words of length $n>0$ in $X^{\pm}$ is

$$\begin{cases}(2k-1)^n+(2k-1), & \text{if } n \text{ is even,}\\ (2k-1)^n+1, & \text{if } n \text{ is odd.}\end{cases}$$

This formula was proved to compute the number of the corresponding presentations of one relator groups. We thank V. Guba for giving us an elegant idea of this proof.

Note that the case where R has length smaller than 8 can be easily considered using only Theorems 5.3 and 5.4. Here we consider a more complicated case, where the length of R is equal to 8. Also we have obtained the classification up to hyperbolicity in the case, where the length of R is equal to 9 (this will be published later).

2. Maps

A *map* M is a finite planar connected and simply connected simplicial 2-complex. Its vertices, edges and faces (2-cells) are denoted by $M(0)$, $M(1)$ and $M(2)$ respectively. The boundary of a face π is denoted by $\partial\pi$ and the boundary of M is denoted by ∂M. The *degree* $d(v)$ of a vertex $v\in M(0)$ is the number of edges of M incident to v (loops are counting twice). The *degree* $d(\pi)$ of a face $\pi\in M(2)$ is the number of vertices $v\in\partial\pi$ with $d(v)\geqslant 3$ (counted with their multiplicity). A vertex is called *interior*, if it does not lie on the boundary of M. A face is called *interior*, if no one of its boundary edges lies on the boundary of M. Note, that an interior face can meet the boundary of M in one or more vertices.

A map M is called a *(p,q)-map* if

(1) $d(\pi) \geqslant p$ for every interior face $\pi \in M(2)$.

(2) $d(v) \geqslant q$ for every interior vertex $v \in M(0)$ of degree $3, 4 \ldots$,

(3) no interior vertex of degree 1 exists in M.

A map M is called a *regular (p,q)-map* if $d(\pi) = p$ and $d(v) = q$ in (1)–(2).

The *radius* of M is

$$\mathrm{rd}(M) = \max_{v \in M(0)} \mathrm{dist}(v, \partial M),$$

where $\mathrm{dist}(v, \partial M)$ is the minimal number of edges, needed to compose a path from v to ∂M.

Theorem 2.1. [2]. *Let M be a (p,q)-map, where $(p,q) \in \{(3,6),(4,4),(6,3)\}$ and let the radius of any regular (p,q)-submap of M is bounded by a constant K. Then*

$$|M(2)| \leqslant 100 q^K p |\partial M|.$$

Thus, the isoperimetric inequality in the class of all such maps with a fixed constant K is linear.

3. Diagrams over a group presentation

Let $G = \langle \mathcal{A} \,|\, \mathcal{R} \rangle$ be a group presentation, where $\mathcal{A}$ is an alphabet, $\mathcal{R}$ is a set of cyclically reduced words over $\mathcal{A}^{\pm}$.

A (van Kampen) *diagram* M over G is a map equipped by a labelling function ϕ from the set of oriented edges of M to the alphabet $\mathcal{A}^{\pm 1}$ such that the following two conditions hold.

(1) If $\phi(e) = a$, then $\phi(e^{-1}) = a^{-1}$.

(2) If π is a face of M and $\partial \pi = e_1 \ldots e_l$ is the boundary cycle of π, where $e_1, \ldots, e_l$ are oriented edges, then $\phi(\partial \pi) = \phi(e_1) \ldots \phi(e_l)$ is a cyclic permutation of R^{ϵ}, where $\epsilon = \pm 1$ and $R \in \mathcal{R}$.

Let v be an occurrence of a vertex in the boundary cycle $\partial \pi$. The boundary path of π starting at v and going clockwise is denoted by $\partial_v \pi$.

A diagram M over the presentation $\langle \mathcal{A} \,|\, \mathcal{R} \rangle$ is called *reduced* if there are no two 2-cells π_1, π_2, whose boundaries contain a common vertex v with $\phi(\partial_v \pi_1) = \phi(\partial_v \pi_2)^{-1}$.

Lemma 3.1. (van Kampen). *A cyclically reduced non-empty word W in the alphabet $\mathcal{A}^{\pm}$ equals 1 in the group $G = \langle \mathcal{A} \,|\, \mathcal{R} \rangle$ if and only if there is a reduced diagram M over G such that $\phi(\partial M) = W$ (graphical equality).*

4. Hyperbolicity of small cancellation groups satisfying the $C(p)\&T(q)$-condition

The presentation $G = \langle \mathcal{A} \,|\, \mathcal{R} \rangle$ satisfies the *small cancellation conditions* $C(p) \,\&\, T(q)$ if any reduced diagram over G is a (p, q)-map. This definition is equivalent to the following one (algorithmically verifiable), given in [7].

Let $\mathcal{R}^*$ be a set of all cyclic permutations of all words from $\mathcal{R}$ and their inverses. A nontrivial word $u \in F(\mathcal{A})$ is called a *piece* (for this presentation), if there exist $r_1, r_2 \in \mathcal{R}^*$, such that $r_1 = uv_1$, $r_2 = uv_2$ for some v_1, v_2, $v_1 \neq v_2$. A group G satisfies the *small cancellation condition* $C(p)$ if no one element of $\mathcal{R}^*$ can be expressed as a product of less than p pieces. A group G satisfies the *small cancellation condition* $T(q)$ if for any $r_1, \dots, r_k \in \mathcal{R}^*$, where $3 \leqslant k < q$ some product $r_i r_{i+1}$ is reduced ($i = 1, \dots, k$, subscripts mod k).

A diagram M over G is called *minimal* if it has minimal number of faces among all diagrams M' over G satisfying $\phi(\partial M) = \phi(\partial M')$. Note that any minimal diagram is reduced, but the converse is not necessarily true.

Theorem 4.1. [2]. *Let $G = \langle \mathcal{A} \,|\, \mathcal{R} \rangle$ be a finitely presented group satisfying a small cancellation condition $C(p) \,\&\, T(q)$ with $(p, q) \in \{(3, 6), (4, 4), (6, 3)\}$. Then G is hyperbolic if and only if there exists a constant K such that the radius of every minimal regular (p, q)-diagram over G does not exceed K.*

Example 1. $G = \langle a, b \,|\, a^3b^3ab \rangle$.

It is straightforward to verify that G is a $(4, 4)$-group. There are only two splittings of the cyclic word $R = a^3b^3ab$ into 4 pieces:

$$a^2 \cdot ab \cdot b^2 \cdot ab, \qquad\qquad a^2 \cdot b^2 \cdot ba \cdot ba,$$

and there are only two splittings of the cyclic word $R^{-1} = b^{-1}a^{-1}b^{-3}a^{-3}$ into 4 pieces:

$$(ab)^{-1} \cdot b^{-2} \cdot (ab)^{-1} \cdot a^{-2}, \qquad\qquad (ba)^{-1} \cdot (ba)^{-1} \cdot b^{-2} \cdot a^{-2},$$

We draw the corresponding labelled squares in Figure 1 (the low two squares are obtained from the upper two by reflection in the vertical axis):

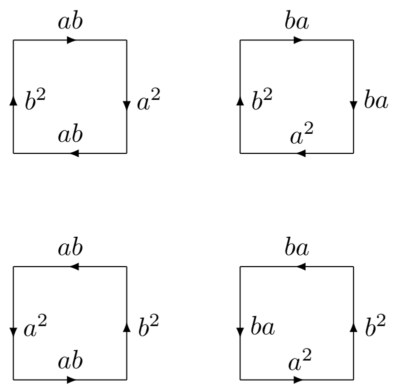

Figure 1

Now we try to paste copies of them (using rotations) to create a large reduced regular $(4, 4)$-diagram. One of the possible diagrams is drawn on Figure 3. Its radius equals to 1.

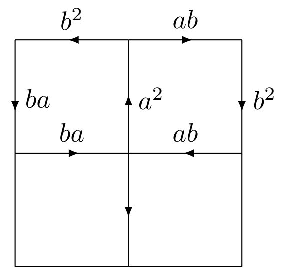

Figure 2

Let us prove that the radius of any reduced regular $(4, 4)$-diagram over G is at most 4. In each square, there is a side labelled by a^2. Taking one of them, there is the unique way to glue another one along the a^2-side to get a reduced diagram (see Figure 2). But then there is impossible to glue two squares from below preserving the reducibility. Thus, the beginning of the side labelled by a^2 must lie on the boundary of any reduced regular

(4, 4)-diagram over G. Hence the radius of any such diagram is at most 4. By applying Theorem 4.1, we conclude that G is hyperbolic.

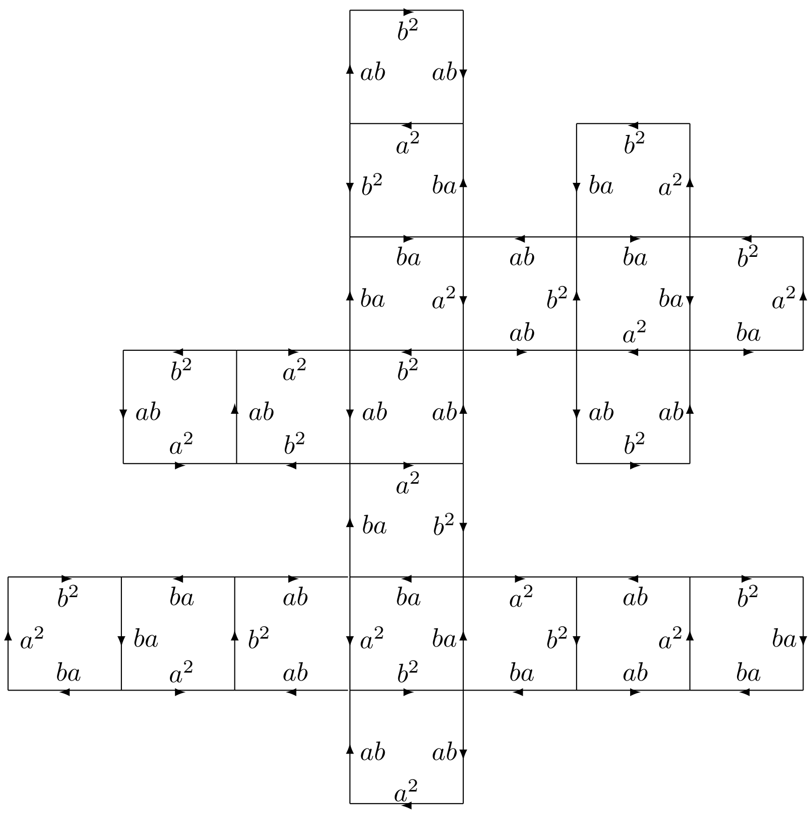

Figure 3

Example 2. $G = \langle a, b \,|\, a^{-3}bab^3 \rangle$.

It is straightforward to verify that G is a (6, 3)-group. There are only 4 splittings of the cyclic word $R = a^{-3}bab^3$ into 6 pieces (the same holds for R^{-1}):

$$\begin{array}{ll} a^{-2}\cdot a^{-1}\cdot b\cdot a\cdot b^2\cdot b, & b^{-1}\cdot b^{-2}\cdot a^{-1}\cdot b^{-1}\cdot a\cdot a^2, \\ a^{-1}\cdot a^{-2}\cdot b\cdot a\cdot b^2\cdot b, & b^{-1}\cdot b^{-2}\cdot a^{-1}\cdot b^{-1}\cdot a^2\cdot a, \\ a^{-2}\cdot a^{-1}\cdot b\cdot a\cdot b\cdot b^2, & b^{-2}\cdot b^{-1}\cdot a^{-1}\cdot b^{-1}\cdot a\cdot a^2, \\ a^{-1}\cdot a^{-2}\cdot b\cdot a\cdot b\cdot b^2, & b^{-2}\cdot b^{-1}\cdot a^{-1}\cdot b^{-1}\cdot a^2\cdot a. \end{array}$$

Thus, we have 8 types of 6-gons, and one can built reduced regular $(6,3)$-diagrams over G using copies of them. Let us prove that the radius of any such diagram M is at most 5.

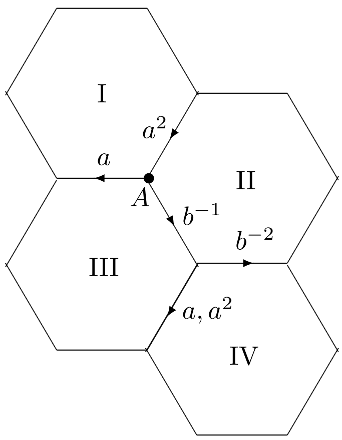

Figure 4

First we note, that each of the 6-gons has a unique a^2-side.

Take an arbitrary 6-gon I of M. Let e be its a^2-side, and let A be the terminal vertex of e. We show that A is at distance at most 1 from the boundary of M. This will immediately imply that $\mathrm{rd}(M) \leqslant 5$.

We may assume that A is an inner vertex. Then there are another two 6-gons, say II and III, around the vertex A. Note that in any 6-gon the sides following immediately after the a^2-side can only have labels: b^{-1}, b^{-2} and a. Therefore one of the two sides following e in $\mathrm{I} \cup \mathrm{II}$ has label a. Without loss of generality we may assume that this side belongs to I (otherwise it belongs to II and we can start from II at the beginning). Then the other side has label b^{-1} (otherwise it would have label b^{-2}, and we would read b^2a on the counterclockwise contour of III, what is impossible).

Thus, after the a^2-side in II we see the b^{-1}-side. Then the next one is the b^{-2}-side. Similarly the side, which follows after b^{-1}-side in III has label a or a^2.

Now, if we assume that the 6-gon IV (adjacent to II and III) exists, we would read b^2a or b^2a^2 on its counterclockwise contour, what is impossible. Therefore the n-gon IV does not exist and so $\mathrm{dist}(A, \partial M) = 1$. By applying Theorem 4.1, we conclude that G is hyperbolic.

Example 3. $G = \langle a, b \,|\, a^{-1}b^{-1}a^3b^3 \rangle$.

It is straightforward to verify that G is a $(6,3)$-group. There are only 4 splittings of the cyclic word $R = a^{-1}b^{-1}a^3b^3$ into 6 pieces (the same holds for R^{-1}):

$$
\begin{array}{ll}
a^{-1} \cdot b^{-1} \cdot a \cdot a^2 \cdot b \cdot b^2, & \quad b^{-2} \cdot b^{-1} \cdot a^{-2} \cdot a^{-1} \cdot b \cdot a, \\
a^{-1} \cdot b^{-1} \cdot a^2 \cdot a \cdot b \cdot b^2, & \quad b^{-2} \cdot b^{-1} \cdot a^{-1} \cdot a^{-2} \cdot b \cdot a, \\
a^{-1} \cdot b^{-1} \cdot a \cdot a^2 \cdot b^2 \cdot b, & \quad b^{-1} \cdot b^{-2} \cdot a^{-2} \cdot a^{-1} \cdot b \cdot a, \\
a^{-1} \cdot b^{-1} \cdot a^2 \cdot a \cdot b^2 \cdot b, & \quad b^{-1} \cdot b^{-2} \cdot a^{-1} \cdot a^{-2} \cdot b \cdot a.
\end{array}
$$

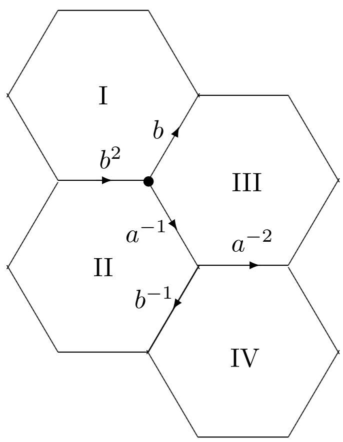

Figure 5

Thus, we have 8 types of 6-gons. As in Example 6, one can prove that the radius of any reduced regular $(6,3)$-diagrams over G is at most 5. By applying Theorem 4.1, we conclude that G is hyperbolic.

5. More tools to establish (non)hyperbolicity

Proposition 5.1. *Any group G given by a one-relator presentation $\langle \mathcal{A} | R^n \rangle$, where $n \geqslant 2$, is hyperbolic.*

Proof. We may assume that R is cyclically reduced. By Newman's spelling theorem [10], if w is a nontrivial reduced word in the alphabet $A^{\pm}$, such that $w =_G 1$, then w contains a subword of $R^{\pm n}$ of length greater than $(n-1)/n$ times the length of R^n. This immediately implies that this presentation has a linear isoperimetric inequality, and hence G is a hyperbolic group.

The next proposition easily follows from Theorems 8.2.A and 3.1.A of [5].

Proposition 5.2. *For any two elements x, y of a torsion-free hyperbolic group the following hold.*

1) *If $x^k = y^k$ for some $k \neq 0$, then $x = y$.*

2) *If $x^{-1} y^k x = y^l$, where y is nontrivial and $k \neq 0$, then $k = l$ and x, y lie in the same cyclic subgroup.*

Below $|R|_a$ denotes the number of occurrences of letters a and a^{-1} in the word R. From proposition 5.2 one can easily deduce the following theorem[1].

Theorem 5.3. *Let $G = \langle a, b \,|\, R \rangle$ be a one-relator group, R a cyclically reduced word over the alphabet $\{a, b\}^{\pm}$. If $|R|_a = 2$, then G is hyperbolic if and only if R is a proper power.*

The following theorem is a partial case of Theorem 3 from [2].

Theorem 5.4. [2]. *Let $G = \langle a, b \,|\, R \rangle$ be a one-relator group, R a cyclically reduced word over the alphabet $\{a, b\}^{\pm}$ with $|R|_a = 3$. Then G is not hyperbolic if and only if R has one of the following forms up to cyclic permutations and taking inverses:*

(1) $R = ab^k ab^l ab^m$ *and for* $n_1 = l - k$ *and* $n_2 = m - k$ *one of* (1a)-(1d) *holds:*

(1a) $\min(|n_1|, |n_2|) = 0$ *and* $\max(|n_1|, |n_2|) > 1$.
(1b) $\min(|n_1|, |n_2|) > 0$ *and* $|n_1| = |n_2| \neq 1$.
(1c) $\min(|n_1|, |n_2|) > 0$ *and* $n_1 = -n_2$.
(1d) $\min(|n_1|, |n_2|) > 0$ *and* $n_1 = 2n_2$ *(or* $n_2 = 2n_1$*).*

(2) $R = ab^k ab^l a^{-1} b^m$ *and either* (2a) *or* (2b) *holds:*
(2a) $|l| = |m|$.
(2b) $l = -2m$ *(or* $m = -2l$*).*

The following theorem is a partial case of Theorem 4 from [2].

[1] This theorem also follows from results of papers [1], [6], and [9]; see also [2].

Theorem 5.5. [2]. *Let $G = \langle a, b \,|\, R\rangle$ be a one-relator group, R a cyclically reduced word over the alphabet $\{a, b\}^{\pm}$ of the form*

$$R = ab^{n_0}ab^{n_1}ab^{n_2}ab^{n_3},$$

where $n_i \neq n_j$ for $i \neq j$. Then G is not hyperbolic if and only if for some $i \in \{0, 1, 2, 3\}$ one has (with subscripts mod 4*) that*

$$n_i + n_{i+2} = n_{i+1} + n_{i+3}.$$

Notation. Let α be an automorphism of F_2 of infinite order. We write $F_2 \rtimes_\alpha \mathbb{Z}$ for the semidirect product determined by the action of α on F_2. This group has the presentation $\langle x, y \,|\, t^{-1}xt = \alpha(x),\ t^{-1}yt = \alpha(y)\rangle$.

Proposition 5.6. *Any semidirect product $F_2 \rtimes_\alpha \mathbb{Z}$ is not a hyperbolic group.*

Proof. Let x, y be a basis of F_2. By a result of Nielsen, $\alpha([x, y]) = u^{-1}[x, y]^{\varepsilon}u$ for some $u \in F_2$ and $\varepsilon \in \{-1, 1\}$. For the semidirect product $G = F_2 \rtimes_\alpha \mathbb{Z}$ with the above notation this means that $t^{-1}[x, y]t = u^{-1}[x, y]^{\varepsilon}u$, that is $[x, y]$ (anti)commutes with tu^{-1}. If G were a hyperbolic group, we would obtain a contradiction to Proposition 5.2. Hence G is not hyperbolic.

Proposition 5.7. *Let G be a group, given by a presentation $\langle a, b \,|\, R\rangle$ with 2 generators and one relator. If G is a free abelian group, then either R is a primitive element of the free group $F(a, b)$, or R is conjugate to $[a, b]$, or to $[a, b]^{-1}$ in $F(a, b)$.*

Proof. The first part is a corollary of a theorem of Whitehead[2]. To prove the second path, one should note that if $G \cong \mathbb{Z}\times\mathbb{Z}$, then the normal closures of $[a, b]$ and R coincide. Hence, by a theorem of Magnus[3], R is conjugate to $[a, b]$, or to $[a, b]^{-1}$ in $F(a, b)$.

Example 4. $G = \langle a, b \,|\, a^{-1}b^{-1}a^2bab^2\rangle$.

Clearly G is a torsion free group. Suppose that G is hyperbolic. First we rewrite the relator of this presentation as $(a^{-1}b^{-1}aba)^2 = b^{-2}$. By Proposition 5.2 we have $a^{-1}b^{-1}aba = b^{-1}$. Now we write this relation in the form $(ab)^2 = a^{-1}(ab)a$. Again by Proposition 5.2, we have $ab = 1$. This implies that $G \cong \mathbb{Z}$, a contradiction with Proposition 5.7. Thus, the group G is not hyperbolic.

Example 5. $G = \langle a, b \,|\, a^{-1}b^{-1}a^{-1}b^{-1}a^2b^2\rangle$.

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[2]**Theorem** [12]. A one-relator group $G = \langle \mathcal{A} \,|\, R\rangle$ is free if and only if R is a primitive element in the free group $F(\mathcal{A})$.

[3]**Theorem** [8]. Let F be a free group and let R_1 and R_2 be two elements of F with the same normal closure. Then R_1 is conjugate to R_2 or to R_2^{-1}.

Clearly G is a torsion free group. Suppose that G is hyperbolic. First we rewrite the relator of this presentation as $[aba, ab^2] = 1$. By Proposition 5.2 there exists $z \in G$ such that $aba = z^k$ and $ab^2 = z^l$ for some k, l. Then $(aba)^l(ab^2)^{-k} = 1$. But $G/G' \cong \mathbb{Z} \times \mathbb{Z}$. This implies, that $2l - k = 0$ and $l - 2k = 0$, and so $k = l = 0$. Hence $aba = 1$ and $ab^2 = 1$. This implies that $b^3 = 1$, a contradiction.

Sometimes we rewrite a relation in new generators.

Example 6. $G = \langle a, b \,|\, a^{-1}ba^{-1}babab\rangle$.

Replacing $b \mapsto ab$, we get the new presentation of the same group: $\langle a, b \,|\, b^2a^2ba^2b\rangle$. This implies that $b^{-2} = (a^2b)^2$. Suppose that G is hyperbolic. Then $b^{-1} = a^2b$, that is $b^{-2} = a^2$. This implies $b^{-1} = a$, and hence b has order 4, what is impossible.

Example 7. $G = \langle a, b \,|\, a^2b^{-1}a^{-1}bab^2\rangle$.

Replacing $b \mapsto ba^{-1}$, we get a new presentation of the same group: $\langle a, b \,|\, a^2b^{-1}a^{-1}b^2a^{-1}b\rangle$. Now we denote $y_i = a^iba^{-i}$, $i \in \mathbb{Z}$. Then G has the presentation $\langle a, y_i (i \in \mathbb{Z}) \,|\, ay_ia^{-1} = y_{i+1},\ y_2^{-1}y_1^2y_0 = 1\rangle$. Thus $y_2 = y_1^2y_0$ and even $y_{i+2} = y_{i+1}^2y_i$ for $i \in \mathbb{Z}$. Using Tietze transformations, we obtain the following presentation of G:

$$\langle y_0, y_1, a \,|\, ay_0a^{-1} = y_1,\ ay_1a^{-1} = y_1^2y_0\rangle.$$

Thus, $G \cong F_2 \rtimes_\alpha \mathbb{Z}$ for some automorphism α of F_2 of infinite order. By Proposition 5.6, the group G is not hyperbolic.

6. The classification

In this section we give the classification up to hyperbolicity of all groups given by a presentation $\langle a, b \,|\, R\rangle$, where R is a cyclically reduced word of length 8 in the alphabet $\{a, b\}^{\pm}$. There are 6564 such presentations. We consider them up to equivalence modulo permutations and inversions of a, b, and modulo cyclic permutations and inversions of R. There are 83 such equivalence classes. Representatives of these classes are listed below. We have grouped them into classes according to small cancellation conditions $C(p) \,\&\, T(q)$ which they satisfy. In each case we found the maximal possible p and q. This was done with the help of a computer program, written by the third author. The condition $C(\infty)$ means that the corresponding word R cannot be slitted into pieces and so it satisfies the condition $C(n)$ for each finite n. The condition $T(\infty)$ means that R satisfies the condition $T(n)$ for each finite n.

Agreements. 1) The sign $+$ (respectively $-$) means that the group is hyperbolic (respectively not hyperbolic).

2) In the middle column, in some cases, we give a short indication why the corresponding group is hyperbolic or not.

If we write there something like $x^{-1}y^kx = y^l$, this means that the arguments use Propositions 5.2 and 5.7 as in Example 4.

If we write there $F_2 \rtimes_\alpha \mathbb{Z}$, this means that the corresponding group can be represented as a split extension of F_2 by $\mathbb{Z}$, and hence it cannot be hyperbolic by Proposition 5.6 (see Example 7).

If we write there nothing, this means that either the group satisfy the condition $C(n)$, $n \geqslant 8$ (in this case it is clearly hyperbolic), or it does not satisfy this condition, but some letter occurs in R exactly 2 or 3 times (in this case we apply Theorem 5.3 or 5.4), or 4 times and R has the form as in Theorem 5.5.

$C(2)\,\&\,T(4)$-groups:

(1) ab^2ab^4		$-$
(2) $abab^2ab^2$		$+$

$C(3)\,\&\,T(3)$-groups:

(1) $ab^{-1}ab^2ab^2$		$-$
(2) $a^{-1}bab^2ab^2$		$+$
(3) $a^{-1}b^{-1}ab^2ab^2$		$-$

$C(3)\,\&\,T(4)$-groups:

(1) a^2babab^2	$(ab)^3 = a^{-1}(ab)^{-1}a$	$-$
(2) $a^2ba^2b^3$	$(a^2b)^2 = b^{-2}$	$-$
(3) $a^{-1}bababab$	$a^2 = (ab)^4$	$-$
(4) $a^{-1}b^{-1}ababab$	$(ab)^3 = b(ab)b^{-1}$	$-$
(5) $a^{-1}b^{-1}a^{-1}babab$	$(ba)^4 = (aba)^2$	$-$
(6) $abab^5$		$-$
(7) $ababab^3$		$-$
(8) $a^2b^2ab^3$		$+$

$C(4)\,\&\,T(3)$-groups:

(1) $aba^{-2}bab^2$	$(bab)^2 = a^2$	$-$
(2) $ab^{-1}a^{-1}ba^2b^2$	$(a^2b^2)^2 = (b^{-1}ab^2)^2$	$-$
(3) $ab^{-1}a^{-2}b^{-1}ab^2$	$F_2 \rtimes_\alpha \mathbb{Z}$	$-$
(4) $a^{-2}b^2a^2b^2$	$a^4 = (a^2b^2)^2$	$-$
(5) $a^{-2}b^{-2}a^2b^2$	$a^2 = (b^{-2}ab^2)^2$	$-$
(6) $aba^{-1}bab^3$		$-$
(7) $ab^{-1}ab^{-1}ab^3$		$-$
(8) $ab^{-1}a^{-1}bab^3$		$-$
(9) $ab^{-2}abab^2$		$+$
(10) $ab^{-2}a^{-1}bab^2$		$-$

(11)	$ab^{-3}ab^3$			$-$
(12)	$a^{-1}b^3ab^3$			$-$
(13)	$a^{-1}babab^3$			$+$
(14)	$a^{-1}b^{-1}abab^3$			$+$
(15)	$a^{-1}b^{-1}ab^{-1}ab^3$			$+$
(16)	$a^{-1}b^{-2}a^{-1}bab^2$			$+$
(17)	$a^{-1}b^{-3}ab^3$			$-$

<u>$C(4)\,\&\,T(4)$-groups:</u>

(1)	aba^2bab^2	$(aba)^2 = b^{-2}$		$-$
(2)	a^3bab^3	$\mathrm{rd}<\mathrm{Const}$ (Ex.1)		$+$
(3)	a^4b^4	$a^4 = b^{-4}$		$-$
(4)	$a^{-1}bab^{-1}abab$	$\xrightarrow{b\mapsto ab}$ $b^2ab^{-1}aba^2$	(Th. 5.5)	$+$
(5)	$a^{-1}ba^{-1}babab$	$\xrightarrow{b\mapsto ab}$ $b^2a^2ba^2b,\quad b^{-2} = (a^2b)^2$	(Ex.6)	$-$
(6)	$a^{-1}b^{-1}aba^{-1}bab$	$(ba^{-1}ba)^2 = (a^{-1}ba)^2$		$-$
(7)	$a^{-1}b^{-1}ab^{-1}abab$	$F_2 \rtimes_\alpha Z$		$-$
(8)	$a^{-1}b^{-1}ab^{-1}a^{-1}bab$	$b^{a^{-1}ba} = b$		$-$
(9)	$a^{-1}b^{-1}a^{-1}b^{-1}abab$	$(ba)^2 = (ab)^2$		$-$
(10)	a^2b^6			$-$
(11)	a^2bab^4			$+$
(12)	a^3b^5			$-$

<u>$C(5)\,\&\,T(3)$-groups:</u>

(1)	$a^2ba^{-1}bab^2$			$+$
(2)	$a^2b^{-1}abab^2$			$-$
(3)	$a^2b^{-1}a^2b^3$	$(ba^{-2})^2 = b^4$		$-$
(4)	$a^2b^{-1}a^{-1}bab^2$	$\xrightarrow{b\mapsto ba^{-1}}$ $a^2b^{-1}a^{-1}b^2a^{-1}b,\quad F_2 \rtimes_\alpha \mathbb{Z}$	(Ex. 7)	$-$
(5)	$ab^{-1}a^{-2}bab^2$	$F_2 \rtimes_\alpha \mathbb{Z}$		$-$
(6)	$a^{-1}ba^2bab^2$			$+$
(7)	$a^{-2}babab^2$	$\xrightarrow{b\mapsto b^{-1}a^{-1}}$ $(bab^3a^3)^{-1}$	$\mathrm{rd}<\mathrm{Const}$ (Ex. 1)	$+$
(8)	$a^{-2}ba^2b^3$	$b^3 = a^{-2}ba^2$		$-$
(9)	$a^{-2}b^{-1}abab^2$	$F_2 \rtimes_\alpha \mathbb{Z}$		$-$
(10)	$a^{-2}b^{-1}a^2b^3$	$b^3 = a^{-2}ba^2$		$-$
(11)	$a^{-1}b^{-1}a^2bab^2$	$(a^{-1}b^{-1}aba)^2 = b^{-2}$	(Ex. 4)	$-$
(12)	$a^{-1}b^{-1}a^{-1}ba^2b^2$	$\xrightarrow{b\mapsto ba^{-2}}$ $(b^{-1}a^2b^{-2}aba)^{-1}$	(Th. 5.5)	$+$
(13)	$a^{-1}b^{-1}a^{-1}b^{-1}a^2b^2$	$[aba, ab^2]$	(Ex. 5)	$-$
(14)	$ab^{-1}ab^5$			$-$
(15)	$ab^{-1}abab^3$			$-$
(16)	$ab^{-1}a^{-1}b^{-1}ab^3$			$-$
(17)	$ab^{-2}ab^4$			$-$

(18) $a^{-1}b^2ab^4$		$-$
(19) $a^{-1}bab^5$		$-$
(20) $a^{-1}ba^{-1}b^2ab^2$		$-$
(21) $a^{-1}b^{-1}ab^5$		$-$
(22) $a^{-1}b^{-1}a^{-1}b^2ab^2$		$-$
(23) $a^{-1}b^{-1}a^{-1}bab^3$		$+$
(24) $a^{-1}b^{-2}ab^4$		$-$
(25) $a^{-1}b^{-2}abab^2$		$-$
<u>$C(6)\,\&\,T(3)$-groups:</u>		
(1) $a^{-3}bab^3$	rd<Const (Ex. 2)	$+$
(2) $a^{-2}b^{-1}a^{-1}bab^2$	$\xrightarrow{a\mapsto ab^{-1}}$ $a^{-1}ba^{-2}bab^2$ (Th. 5.5)	$+$
(3) $a^{-1}b^{-1}a^3b^3$	rd<Const (Ex. 3)	$+$
(4) $ab^{-1}a^2b^4$		$+$
(5) $ab^{-2}a^2b^3$		$+$
(6) $a^{-2}b^2ab^3$		$+$
(7) $a^{-2}bab^4$		$+$
(8) $a^{-1}b^{-1}a^2b^4$		$+$
(9) $a^{-1}b^{-2}a^2b^3$		$+$
<u>$C(8)\,\&\,T(4)$-groups:</u>		
(1) $(a^2b^2)^2$		$+$
(2) $(a^{-1}bab)^2$		$+$
(3) $(a^{-1}b^{-1}ab)^2$		$+$
<u>$C(\infty)\,\&\,T(4)$-groups:</u>		
(1) ab^7		$+$
(2) $(ab^3)^2$		$+$
<u>$C(\infty)\,\&\,T(\infty)$-groups:</u>		
(1) $(ab)^4$		$+$
(2) b^8		$+$

7. The number of cyclically reduced words of a given length in a given alphabet

Let X be a nonempty finite set of cardinality k. It is well known, that the number of reduced words of length $n>0$ over the alphabet $X^{\pm}$ is

$$(2k)(2k-1)^{n-1}.$$

The following proposition gives an explicit formula for the number of cyclically reduced words. This formula can be also proven by constructing an automaton, which accepts all cyclically reduced words in $X^{\pm}$. However, further computations with the corresponding matrix are routine. Therefore we present here a short an elegant combinatorial proof, based on an idea of V. Guba.

We use this proposition to justify that the number of presentations $\langle a, b \mid R\rangle$, where R is a cyclically reduced word of length 8 is 6564 (see the table in Section 8).

Proposition 7.1. *Let X be a nonempty finite set of cardinality k. The number of cyclically reduced words of length $n > 0$ over the alphabet $X^{\pm}$ is*

$$\begin{cases} (2k-1)^n + (2k-1), & \text{if } n \text{ is even,} \\ (2k-1)^n + 1, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let A_n be the set of all reduced words of length n in the alphabet $X^{\pm}$, and let B_n (respectively C_n) be the subset of A_n, consisting of cyclically reduced (respectively not cyclically reduced) words. Clearly, $|A_n| = |B_n| + |C_n|$. For every letter $x \in X^{\pm}$ we denote by $B_n(x)$ and by $C_n(x)$ the subsets of B_n and of C_n, consisting of words starting from x.

Lemma. $|C_n| = |B_{n-1}| - 2k$.

Proof. It is enough to prove that $|C_n(x)| = |B_{n-1}(x)| - 1$ for any letter $x \in X^{\pm}$. Note that any word from $C_n(x)$ ends on x^{-1}. Define a map $\varphi : C_n(x) \to B_{n-1}(x) \setminus \{x^{n-1}\}$ by the following rule. If $w \in C_n(x)$ has the form $w = ux^{-l}$, where u does not end on x^{-1}, then $\varphi(w) = ux^{l-1}$. It is easy to show that this map is a bijection.

Now, $(2k)(2k-1)^{n-1} = |A_n| = |B_n| + |C_n| = |B_n| + |B_{n-1}| - 2k$, and the proof of Proposition follows by induction on n.

8. The final table

In the following table we summarize an information on presentations $\langle a, b \mid R\rangle$, where R is a cyclically reduced word of length 8. "Total" means the total number of such presentations. "Total.M" means the number of such presentations modulo permutations and inversions of a, b, and modulo cyclic permutations and inversions of R. The abbreviations "Hyp" and "nHyp" are reserved for the hyperbolic and non-hyperbolic presentations, respectively. Rows are designed for specializing the numbers in cases $C(p) \,\&\, T(q)$.

$C(p)\,\&\,T(q)$	Total	Total.M	Total.M.Hyp	Total.M.nHyp
(2,4)	128	2	1	1
(3,3)	320	3	1	2
(3,4)	576	8	1	7
(4,3)	1328	17	5	12
(4,4)	672	12	3	9
(5,3)	2304	25	5	20
(6,3)	1088	9	9	0
(8,4)	40	3	3	0
$(\infty, 4)$	96	2	2	0
(∞, ∞)	12	2	2	0
	6564	83	32	51

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Oleg Bogopolski, Institute of Mathematics of SBRAS, Novosibirsk, Russia, and Fachbereich Mathematik, Universität Dortmund, Germany,
E-mail: groups@math.nsc.ru

Nikolai Buskin, Novosibirsk State University, Russia,
E-mail: buskin1983@mail.ru

Alexander Buturlakin, Institute of Mathematics of SBRAS, Novosibirsk, Russia, *E-mail*: buturlakin@gorodok.net